USE OF SINGLE-POINT VELOCITY PROBABILITY DISTRIBUTIONS IN DESCRIBING TURBULENT FLOWS

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A semiemprical equation for the single-point velocity probability distribution in turbulent flows is suggested and analyzed. The inertia forces in the equations are exactly expressed in terms of probability distributions. The remaining terms, related to pressure forces and viscosity, are not accurately expressed in terms of the true probability distributions, and semiempirical expressions are used to approximate them. Some arbitrariness is generated in approximating the pressure term. It is selected from the coincidence condition of the corresponding terms in the equations for the second moments, following from the equation for the probability distributions and used in the available semiempirical theories.

One of the main features of the equation obtained is its nonlocality, which agrees, at least qualitatively, with contemporary concepts on laws of turbulent transport. The energy dissipation velocity, the fundamental characteristic of turbulence, plays an important role in the equation.

The solution of the equation is a normal distribution in the flow regions where the energy balance of turbulent motion reduces basically to generation and dissipation. This conclusion is in satisfactory qualitative agreement with experimental data in a logarithmic layer.

1. Basic Equations. The exact nonclosed equation for the single-point probability distribution, obtained in [1-4] from the Navier–Stokes equations, is

$$\frac{\partial P}{\partial t}+u_\alpha\frac{\partial P}{\partial x_\alpha}-\frac{\partial \pi_\alpha}{\partial x_\alpha}+\frac{\partial^2}{\partial u_\alpha \partial u_\beta}\langle\varepsilon_{\alpha\beta}\rangle_u P-\nu\frac{\partial^2 P}{\partial x_\alpha \partial x_\alpha}=0, \tag{1.1}$$

where $P(\mathbf{u}, \mathbf{x}, t)$ is the velocity probability distribution; t, time; x_k, point coordinates ($k=1, 2, 3$); u_k, hydrodynamic velocity; $\varepsilon_{ij}=\nu[(\partial u_i/\partial x_\alpha)(\partial u_j/\partial x_\alpha)]$, instant velocity tensor of energy dissipation; ν, molecular viscosity coefficient; $\pi_k=\langle\partial p/\partial x_k\rangle_u P$; p, kinematic pressure; and the symbols $\langle\,\rangle$, $\langle\,\rangle_u$ denote, respectively, total (unconditional) averaging and averaging for a given value of **u**. Repeated subscripts denote henceforth summation from 1 to 3.

The first two terms in (1.1) describe inertial forces, the third – pressure forces, and the last two – viscous forces. It is important that the inertia forces are exactly expressed in terms of the true probability distribution. This is the main advantage of using the probability distribution rather than available semiempirical theories for second moments. As is well known, the third moment approximation is quite complicated in these equations. Pressure and viscosity forces are not exactly expressed in terms of P(u), and therefore, as well as in the semiempirical theories for second moments, the approximation of these terms requires the inclusion of nonrigid considerations. The analogy with kinetic theory was used with this purpose in [5-8]. In [8] one finds general considerations on the closure of these equations for finite-dimensional probability distributions. In the present work we use a different variant of closure, based on the results of [3, 4].

Consider first viscous forces. For large Reynolds numbers and outside regions immediately adjacent to the walls (conditions assumed to be satisfied in what follows, the last term in (1.1), describing diffusion of averaged characteristics due to molecular viscosity, can be omitted. Following [3, 4], for the quantity $\langle\varepsilon_{ij}\rangle_u$ we adopt the hypothesis

$$\langle\varepsilon_{ij}\rangle_u=(1/3)\langle\varepsilon\rangle\delta_{ij},\ \varepsilon=\varepsilon_{\alpha\alpha}, \tag{1.2}$$

where $\langle\varepsilon\rangle$ is the dissipation velocity of turbulence energy.

The quantity $\langle\varepsilon_{ij}\rangle_u$ was experimentally measured [9], where it was established that (1.2) is a good approximation, and it has been noted that hypothesis (1.2) is valid only for a completely turbulent fluid. Consequently,

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account of the intermittance somewhat changes the equation for the probability distribution. The equations for this case were obtained in [10]. According to [9], the effect of intermittance on the velocity probability distribution is weak, therefore we use (1.2) at all flow points.

We turn now to approximating the functions π_k. We separate in them explicitly the gradient of the mean pressure $\pi_k = (\partial\langle p\rangle/\partial x_k)P + \delta\pi_k$.

The functions $\delta\pi_k$ are related only to pressure fluctuations and characterize three processes: 1) energy redistribution of turbulent motion between the different components of the velocity fluctuation vector due to the nonlinear interaction of velocity fluctuations (the contribution of this process to $\delta\pi_k$ is denoted by $\pi_k^{(1)}$); 2) energy redistribution between different directions, but due to turbulence deformation in the shear average flow (the corresponding contribution is denoted by $\pi_k^{(2)}$); 3) energy transport in space. In the equation of turbulent energy the latter process is described by the expression $\partial\langle p'v_\alpha\rangle/\partial x_\alpha$, $p' = p - \langle p\rangle$, $v_k = (u_k - \langle u_k\rangle)$, the components of the velocity fluctuation vector. The contribution of this process to $\delta\pi_k$ is denoted by $\pi_k^{(3)}$. Thus, $\delta\pi_k = \pi_k^{(1)} + \pi_k^{(2)} + \pi_k^{(3)}$. The third component is the most important in free turbulent flows. It has an essentially nonlocal character, which generates serious difficulties in approximating it. At the same time, according to experimental data (see, e.g., [11]), the contribution of the third process to the total energy balance of turbulent motion can be neglected in a number of cases. In first approximation it is further assumed that $\pi_k^{(3)} = 0$.

Following [12], we assume that the relation between $\delta\pi_k$ and P is described by a differential relation containing the derivatives of P with respect to u_k, but not of higher than first order. The last restriction follows from the fact that in the opposite case the order of Eq. (1.1) is higher than second, and the qualitative structure of its solution will be determined by the term $\partial\delta\pi_\alpha/\partial u_\alpha$, and not $(1/3)\langle\varepsilon\rangle\partial^2 P/\partial u_\alpha^2$, which does not correspond to contemporary concepts on the important role of the rate of energy dissipation. Furthermore, the relation between $\delta\pi_k$ and P must be such that the following conditions be satisfied identically:

$$\left\langle\frac{\partial p'}{\partial x_k}\right\rangle = \int \delta\pi_k d^3\mathbf{u} = 0,\quad \left\langle v_\alpha \frac{\partial p'}{\partial x_\alpha}\right\rangle = \left\langle\frac{\partial p' v_\alpha}{\partial x_\alpha}\right\rangle - \left\langle p' \frac{\partial v_\alpha}{\partial x_\alpha}\right\rangle = \int v_\alpha \delta\pi_\alpha d^3\mathbf{u} = \int v_\alpha \pi_\alpha^{(3)} d^3\mathbf{u} = 0.$$

Computational requirements do not allow a unique relation between $\delta\pi_k$ and P. The ultimate choice of expressions for $\delta\pi_k$ is made from the condition that the expression for the correlation $\langle p'(\partial v_i/\partial x_j + \partial v_j/\partial x_i)\rangle$ following from it coincide with that used in available semiempirical theories for second moments (see, e.g., [13]). We then have (the expression for $\pi_k^{(1)}$ was earlier derived in [6])

$$\delta\pi_k = \pi_k^{(1)} + \pi_k^{(2)},\quad \pi_k^{(1)} = T^{-1}\left(v_k P + \sigma^2 \frac{\partial P}{\partial v_k}\right), \tag{1.3}$$

$$\pi_k^{(2)} = A\frac{\partial U_k}{\partial x_\alpha}\sigma^2\frac{\partial P}{\partial v_\alpha} + \frac{\partial U_\alpha}{\partial x_\beta}\left(D_1 T_{k\beta}\frac{\partial P}{\partial v_\alpha} + D_2 T_{k\alpha}\frac{\partial P}{\partial v_\beta} + C T_{\alpha\beta}\frac{\partial P}{\partial v_k}\right),$$

$$D_1 + D_2 + 3C = 0,\quad U_k = \langle u_k\rangle,$$

where $T_{ij} = \langle v_i v_j\rangle$ is the Reynolds stress tensor; $\sigma^2 = T_{\alpha\alpha}/3$; $T = R^{-1}\langle\varepsilon\rangle^{-1}\sigma^2$, time scale of trubulence; and R, A, D_1, D_2, and C, empirical constants. They can be related with the second moments in a logarithmic layer, writing out the latter equations (for this Eq. (1.1) must be multiplied by $v_i v_j$ and integrated over v) and using the adopted approximations for $\delta\pi_k$ (1.3), (1.2) (as well known [11], in a logarithmic layer energy diffusion and convection are negligibly small). After simple calculations, we obtain

$$D_1 = 1 - R(2\langle u^2\rangle + \langle v^2\rangle - 3\sigma^2)/\sigma^2,\quad D_2 = -R(\langle u^2\rangle + 2\langle v^2\rangle - 3\sigma^2)/\sigma^2,$$

$$A = [\langle v^2\rangle(1 - D_1) - \langle u^2\rangle D_2]/\sigma^2 - 2Ru_*^4/\sigma^4,\quad C = -(D_1 + D_2)/3,$$

where $v_1 = u$, $v_2 = v$, $v_3 = w$, and u_* is the friction velocity.

In what follows it is convenient to consider the probability distribution for the velocity fluctuation vector $\mathbf{v}$. We also denote this distribution by P(v). An equation for P(v) is obtained from (1.1) after transforming to the new variable $\mathbf{v} = \mathbf{u} - \mathbf{U}$. Using the averaged momentum equation

$$DU_i/Dt = -\partial\langle p\rangle/\partial x_i - \partial T_{i\alpha}/\partial x_\alpha, \tag{1.4}$$

$$D/Dt = \partial/\partial t + U_\alpha \partial/\partial x_\alpha$$

and adopting the approximations for $\langle\varepsilon_{ij}\rangle_u$ and $\delta\pi_k$, we obtain

$$\frac{DP}{Dt} + v_\alpha\frac{\partial P}{\partial x_\alpha} + \left(\frac{\partial T_{\alpha\beta}}{\partial x_\beta} - v_\alpha\frac{\partial U_\beta}{\partial x_\alpha} - T^{-1}v_\alpha\right)\frac{\partial P}{\partial v_\alpha} = D_{\alpha\beta}\frac{\partial^2 P}{\partial v_\alpha \partial v_\beta} + 3T^{-1}P, \tag{1.5}$$

$$D_{ij} = \left(T^{-1}\sigma^2 + C\frac{\partial U_\alpha}{\partial x_\beta}T_{\alpha\beta} - \frac{1}{3}\langle\varepsilon\rangle\right)\delta_{ij} + \frac{1}{2}A\sigma^2\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right) +$$

$$+\frac{1}{2}T_{i\beta}\left(D_1\frac{\partial U_j}{\partial x_\beta}+D_2\frac{\partial U_\beta}{\partial x_j}\right)+\frac{1}{2}T_{j\beta}\left(D_1\frac{\partial U_i}{\partial x_\beta}+D_2\frac{\partial U_\beta}{\partial x_i}\right),\quad D_{ij}=D_{ji}.$$

Here D_{ij} is the diffusion coefficient tensor in phase space. Equations (1.4), (1.5) must be supplemented by the continuity equation of the average velocity field div **U** = 0.

The turbulence energy dissipation rate $\langle\varepsilon\rangle$ appears in (1.5) as an external parameter, therefore in determining it, it is necessary to use corresponding experimental data or to solve a semiempirical equation. The latter is [13]

$$\frac{D\langle\varepsilon\rangle}{Dt}=\frac{2\langle\varepsilon\rangle}{3\sigma^2}(C_{\varepsilon1}g-C_{\varepsilon2}\langle\varepsilon\rangle)+C_\varepsilon\frac{\partial}{\partial x_\alpha}\left(\frac{3\sigma^2}{2\langle\varepsilon\rangle}T_{\alpha\beta}\frac{\partial\langle\varepsilon\rangle}{\partial x_\beta}\right), \tag{1.6}$$
$$g=-\langle v_\alpha v_\beta\rangle\frac{\partial U_\alpha}{\partial x_\beta},\quad C_{\varepsilon1}=C_{\varepsilon2}-k^2(1.5\sigma^2)^2\langle v^2\rangle C_\varepsilon u_*^{-6}.$$

Here $C_\varepsilon C_{\varepsilon1}$, $C_{\varepsilon2}$ are empirical constants (the relation between them is obtained by considering a logarithmic layer), and k is the Karman constant.

The properties of Eq. (1.5) depend essentially on the sign definiteness of the matrix D_{ij} and on the direction of the vector **v**, since these parameters determine the direction of information transport in phase space. The direction of **v** is known ahead of time, but the sign definiteness of the matrix D_{ij} depends, generally speaking, on the coordinates x_k. The surfaces on which the sign definiteness of the matrix D_{ij} changes can, in principle, be singular. In the given case, however, since D_{ij} is independent of **v**, there exist no restrictions on the solution on the surfaces mentioned.

Based on Eq. (1.5), we analyze the probability distribution for homogeneous turbulence ($\langle u_k\rangle = 0$). In this case the equation is

$$\partial P/\partial t-T^{-1}\partial v_kP/\partial v_k=(R-1/3)\langle\varepsilon\rangle\,\partial^2P/\partial v_\alpha^2. \tag{1.7}$$

According to the estimate given in Sec. 2 $(R-1/3)>0$, and, consequently, Eq. (1.7) is a parabolic equation with a positive diffusion coefficient. Assuming $\delta\pi_k = 0$, as was done in [3, 4], Eq. (1.7) transforms to a parabolic equation with negative diffusion coefficients, for which, as well known, the Cauchy problem is incorrect. Thus, account of π_k regularizes the problem.

The general solution of Eq. (1.7) is described by the relation

$$P(\mathbf{v},t)=\int G(\mathbf{v},\mathbf{v}_0,t,t_0)P(\mathbf{v}_0,t_0)\,d^3\mathbf{v}_0, \tag{1.8}$$
$$G=(2\pi)^{-3/2}(\langle u^2\rangle_\delta\langle v^2\rangle_\delta\langle w^2\rangle_\delta)^{-1/2}\exp\left\{-\frac{1}{2}\left[\frac{(u-u_0\varphi)^2}{\langle u^2\rangle_\delta}+\frac{(v-v_0\varphi)^2}{\langle v^2\rangle_\delta}+\frac{(w-w_0\varphi)^2}{\langle w^2\rangle_\delta}\right]\right\},\quad \langle v_i^2\rangle_\delta=\langle v_i^2\rangle-\langle v_i^2\rangle_0\varphi^2,$$
$$\varphi=\exp\left[-R\int_{t_0}^{t}\sigma^{-2}\langle\varepsilon\rangle\,dt\right],\quad \frac{1}{2}\frac{d\langle v_i^2\rangle}{dt}=-T^{-1}\langle v_i^2\rangle+\left(R-\frac{1}{3}\right)\langle\varepsilon\rangle,\ \langle v_i^2\rangle|_{t=t_0}=\langle v_i^2\rangle_0.$$

It follows from (1.8) that the normal distribution retains its form during the process of turbulence degeneration. As is easily seen, for the normal distribution $\delta\pi_k = 0$. In homogeneous isotropic turbulence the normal distribution is naturally bounded for all finite t by the solution of Eq. (1.7) for $\delta\pi_k = 0$ [3, 4], which is related to an inverse parabolic type of equation in this case.

The most important conclusion following from (1.8) is that for $t\to\infty$ the main term in the asymptotic expansion of the probability distribution is described by the self-similar dependence $(\langle\varepsilon\rangle t)^{-3/2}F[|\mathbf{u}|\times(\langle\varepsilon\rangle t)^{-1/2}]$, where F is the isotropic normal distribution, and $\delta\pi_k$ is related with the following terms of this expansion. In connection with the relation obtained we note that the equation suggested in [5] does not possess this important property, and information on the initial distribution occurs in the main term at all times.

We mention a simple solution of Eq. (1.5) for the case in which the turbulence energy balance basically reduces to creation and dissipation. All terms with spatial derivatives can then be neglected in (1.5). By direct substitution it can be verified that in this approximation the solution of Eq. (1.5) is a normal distribution, whose moments are related to the equations for the second moments with omitted convection and diffusion terms (to simplify the calculations it is convenient to transform to characteristic functions). The result obtained is in qualitative satisfactory agreement with experimental data in a logarithmic layer [14, 15].

2. Steady-State Flow in a Channel. Consider a steady-state turbulent flow in a planar channel, whose walls coincide with the walls y = 0 and y = H, while the mean velocity components are $\langle u_1\rangle = U(y)$, $\langle u_2\rangle = \langle u_3\rangle = 0$. Equations (1.4)–(1.6) acquire the following form in the given case

$$0 = -\partial\langle p\rangle/\partial x - d\langle uv\rangle/dy,\ 0 = -\partial\langle p\rangle/\partial y - d\langle v^2\rangle/dy; \quad (2.1)$$

$$v\partial P/\partial y + (d\langle uv\rangle/dy - vdU/dy - T^{-1}u)\partial P/\partial u + \quad (2.2)$$
$$+ (d\langle v^2\rangle/dy - T^{-1}v)\partial P/\partial v - T^{-1}w\partial P/\partial w = D_{\alpha\beta}\partial^2 P/\partial v_\alpha \partial v_\beta + 3T^{-1}P;$$

$$\frac{2\langle\varepsilon\rangle}{3\sigma^2}(C_{\varepsilon 1} g - C_{\varepsilon 2}\langle\varepsilon\rangle) + C_\varepsilon \frac{d}{dy}\left(\frac{3\sigma^2}{2\langle\varepsilon\rangle}\langle v^2\rangle\frac{d\langle\varepsilon\rangle}{dy}\right) = 0. \quad (2.3)$$

The nonvanishing components of the tensor D_{ij} equal to the expressions

$$D_{11} = \left(R - \frac{1}{3}\right)\langle\varepsilon\rangle - (D_1 + C)g,\quad D_{22} = \left(R - \frac{1}{3}\right)\langle\varepsilon\rangle - (D_2 + C)g,$$
$$D_{33} = \left(R - \frac{1}{3}\right)\langle\varepsilon\rangle - Cg,\quad D_{12} = D_{21} = \frac{1}{2}\frac{dU}{dy}(D_1\langle v^2\rangle + D_2\langle u^2\rangle + A\sigma^2).$$

The shear stress distribution $\langle uv\rangle$, appearing in (2.2), (2.3), and in the expressions for the components of the tensor D_{ij}, are found from (2.1):

$$\langle uv\rangle = u_{*0}^2[(1+\omega)y/H - 1],\quad \omega = (u_{*1}/u_{*0})^2. \quad (2.4)$$

The subscripts 0 and 1 in (2.4) refer to different channel walls. The velocity gradient dU/dy in (2.2) (the very velocity defect) is determined from the condition $I_u = \int uPd_3u = 0$, following from the definition of the probability distribution of the velocity fluctuations. To show this we multiply (2.2) by I, u, and v successively, and integrate. As a result we obtain the relations

$$\frac{dI_v}{dy} = 0,\quad \frac{dI_{uv}}{dy} - I_0\frac{d\langle uv\rangle}{dy} + \frac{dU}{dy}I_v = -\frac{1}{T}I_u,$$
$$\frac{d\langle v^2\rangle}{dy} - I_0\frac{d\langle v^2\rangle}{dy} = -\frac{1}{T}I_v,\quad I_v = \int vPd^3\mathbf{u},\quad I_0 = \int Pd^3\mathbf{u},$$
$$I_{uv} = \int uvPd^3\mathbf{u}.$$

Since (by the boundary conditions) $I_v = 0$ for $y = 0$, $y = H$, it follows from the first and third relations that $I_v = 0$, $I_0 = 1$, $0 \le y \le H$. Only the condition $I_u = 0$ remains nontrivial, allowing to determine dU/dy (we also note that for $I_u = 0$ the integral I_{uv} identically equals the expression for $\langle uv\rangle$ (2.4)).

To solve Eq. (2.2) it is necessary to assign the probability distribution at y = 0 and y = H. Since there exists local equilibrium in a logarithmic layer between turbulence energy dissipation and creation, according to the results of Sec. 1 the probability distribution is normal at y = 0 and y = H. From the existence condition of moments of any finite order we also have

$$\lim|\mathbf{v}|^k P = 0,\ |\mathbf{v}| \to \infty \quad \text{for any} \quad k > 0.$$

The probability distribution P depends on four variables, causing substantial difficulties in computational possibilities of numerical solution of Eq. (2.2). The specific structure of Eq. (2.2) makes it possible to avoid this situation. It turns out that (2.2) is equivalent to an infinite system of equations for functions of lower dimensionality. The first four equations of this system are sufficient for determining the mean velocities and second moments. These equations, as well as (2.2), are nonlinear integrodifferential equations, and all remaining ones are linear (it is interesting that a similar situation is also encountered in the kinetic theory of gases in considering model equations; see, for example, [16]). All this is also applicable to the semiempirical equation for the probability distribution obtained in [5], which was earlier used [17] in calculating the planar Couette flow, when $|\langle u, v\rangle| = u_*^2 = \text{const}$ (the solution of (2.2) is a normal distribution in this case).

The first four functions are related to the probability distribution by the following relations:

$$P_2(v) = \int Pdudw,\quad J(v) = \int uPdudw = \langle u\rangle_v P_2,$$
$$H_1(v) = \int u^2 Pdudw = \langle v^2\rangle_v P_2,\quad H_3(v) = \int w^2 Pdudw = \langle w^2\rangle_v P_2.$$

Integrating (2.2), multiplying by $u^l w^m$ ($l = 0$, $m = 0$; $l = 1$, $m = 0$; $l = 2$, $m = 0$; $l = 0$, $m = 2$), the following equations are obtained

$$L(P_2) = T^{-1}P_2,\ L(J) = -S_1\partial P_2/\partial v + S_2 P_2,$$
$$L(H_1) = -T^{-1}H_1 - 2S_1\partial J/\partial v + 2S_2 J + S_3 P_2, \quad (2.5)$$

$$L(H_3) = -T^{-1}H_3 + S_4 P_2,\ S_1 = dU/dy[D_1\langle v^2\rangle + D_2\langle u^2\rangle + A\sigma^2],$$
$$S_2 = (d\langle uv\rangle/dy - vdU/dy),\ S_3 = 2\langle\varepsilon\rangle[R - 1/3 - g(D_1 + C)/\langle\varepsilon\rangle],$$

$$S_1 = 2\langle\varepsilon\rangle\left[R - \frac{1}{3} - gC\langle\varepsilon\rangle\right], \quad L = v\frac{\partial}{\partial y} + \left(\frac{d\langle v^2\rangle}{dy} - T^{-1}v\right)\frac{\partial}{\partial v} - D_{22}\frac{\partial^2}{\partial v^2},$$

$$\langle u^2\rangle = \int H_1\,dv, \quad \langle v^2\rangle = \int v^2 P_2\,dv, \quad \langle w^2\rangle = \int H_3\,dv.$$

The system of equations (2.3), (2.5), supplemented by the condition determining the velocity gradient $I_u = \int uPd^3\mathbf{v} = \int J dv = 0$ is closed. The moments, not expressed in terms of the functions introduced, are found from other equations in the system, but after solving Eqs. (2.3), (2.5). To determine $\langle u^3\rangle$, for example, it is necessary to introduce the function $\int u^3 P du dw$, whose equation, as mentioned above, is linear.

To choose an algorithm for numerical solution of Eq. (2.5) it is important that the coefficient in front of the derivative with respect to the coordinate y change sign. Consequently, despite the fact that (2.5) is a parabolic equation, both directions in the coordinate y are equally valid. This is explained by the two boundary conditions in the coordinate y (the theory of equations of this type has been extensively developed in recent years; see, for example, [18]).

For numerical solution of Eq. (2.5) we used a scheme with counter flow differences (with second order approximation in the internal region), making it possible to account quite simply for the equal justification in the coordinate y noted above. A similar problem for the concentration probability distribution was earlier solved in [19]. Unlike [19], in the given case the diffusion coefficient D_{22} can, generally speaking, change sign for several y values, which must be taken into account in the finite-difference approximate equations. The nonlinear system of finite-difference equations thus obtained was solved by an iteration method.

In carrying out the calculations the system (2.3), (2.5) was nondimensionalized: The dimensional velocity was divided by u_{*0}, and the length by H. The boundary condition corresponding to $|v| \to \infty$ became $|v|_m = 3\langle v^2\rangle_0^{1/2}$. In the case $\omega = 1$, due to the symmetry of the problem a solution was sought in the interval $0 \le y/H \le 0.5$. The difference grid in the variable v was chosen uniform, and in the coordinate y – nonuniform, with a condensation near the walls, as well as for the case $\omega = 0.2$ and near the point of vanishing shear ($y_0/H = 0.83$). The number of sites in the y coordinate was 51. For $\omega = 1$ the calculations were carried out with two grid steps in v, $\Delta v = |v|_m/45$ and $\Delta v = |v|_m/90$, and practically coinciding values between the results were obtained. For $\omega = 0.2$ the calculations were performed with $\Delta v = |v|_m/45$.

For the empirical constants in the calculations we chose the following values: $R = 0.8$ based on experiment [20], $C_{\varepsilon 2} = 2$; $C_\varepsilon = 0.13$ by the recommendations of [22], and $k = 0.41$, the standard value of the Karman constant. The constants A, D_1, D_2, C, and $C_{\varepsilon 1}$ were assigned, starting from experimental data, for the second moments in a logarithmic layer obtained in [14].

It must be noted that in all calculations performed the diffusion coefficient D_{22} is positive in the steady-state solution, though in the iteration process it changes sign multiply.

Figures 1 and 2 show a comparison between calculations and experimental data for the mean velocities and second moments in symmetric ($\omega = 1$) [14] and nonsymmetric channels ($\omega = 0.2$) [22], respectively (1 – $(U_M - U)/u_*$, where U_M is the maximum velocity value, and 2 – $\langle u^2\rangle$; 3 – $\langle v^2\rangle$; 4 – $\langle w^2\rangle$). It is seen that in the calculations one quantitatively reproduces the difference in positions of vanishing shear points ($y_0/H = 0.83$) and the points of vanishing velocity gradient ($y_m/H = 0.73$) for a nonsymmetric channel (which leads to an effect

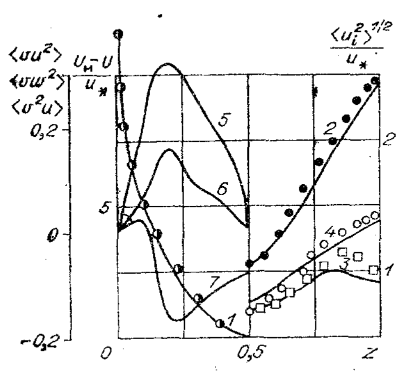

Fig. 1

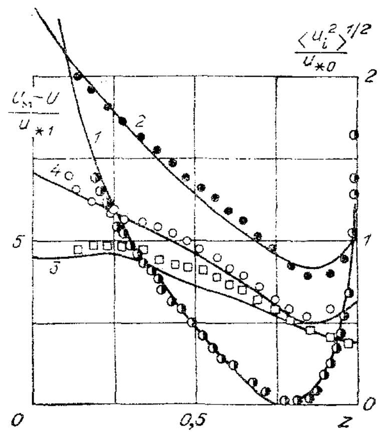

Fig. 2

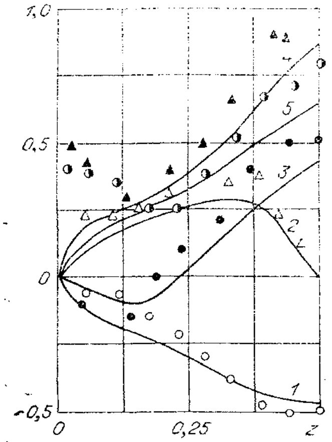

Fig. 3

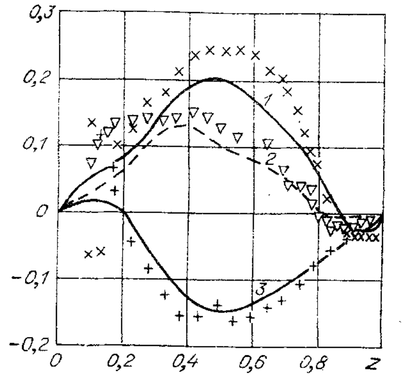

Fig. 4

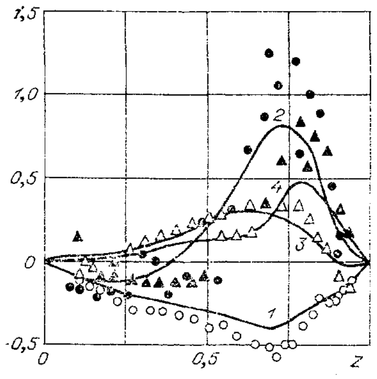

Fig. 5

of "negative" viscosity). This effect was obtained in an earlier calculation [21] by means of an equation for the shear stress $\langle uv^2 \rangle$, in which the third moment was approximated by the gradient $\langle uv^2 \rangle \sim -(\sigma^4/\langle \varepsilon \rangle) d\langle uv \rangle/dy$. We note that the calculation of all second moments in available semiempirical equations for channel flows has, obviously, not yet been carried out, and at the very least there exist no corresponding data in the literature.

We turn now to results of calculating moments of third and fourth order and their comparison with experimental data. Figure 1 shows results of calculating third moments in a symmetric channel (5 – $\langle vu^2 \rangle$, 6 – $\langle vw^2 \rangle$, 7 – $\langle v^2 u \rangle$). Corresponding experimental data were not found in the literature. The results of comparing the calculated and experimental data for the symmetric channel [14] are given in Fig. 3 for the asymmetry and excess coefficients (1 – A_u; 2 – A_v; 3 – E_u; 4 – E_v; 5 – E_w, where A and E are, respectively, the asymmetry and excess coefficients, and the meaning of the lower subscript is that of the accepted notation for the velocity fluctuations). Figures 4 and 5 show the calculated and experimental data in a nonsymmetric channel [22] (on Fig. 4, 1 – $\langle vu^2 \rangle$; 2 – $\langle vw^2 \rangle$; 3 – $\langle v^2 u \rangle$; and on Fig. 5 the notation coincides with that of Fig. 3). The experimental data on Figs. 1-3 from [14] were obtained for Reynolds number $Re = U_p H/2\nu = 230{,}000$ (U_p is the mean velocity, determined from the flow rate), and from [22] – for $Re = U_M H/2\nu = 56{,}000$, and the data on Figs. 4 and 5 were taken from [22] for Re = 36,500.

Analysis of Figs. 1-5 shows that as a whole the coincidence of calculated and experimental data can be assumed quite satisfactory. The data presented on moments, as well as the analysis of the calculated probability distributions, show that in the problem under consideration they are quite close to normal. At the same time, a deviation of the distribution from normal law is important, since only this deviation causes a redistribution of turbulence energy in y. The results of comparing calculated and experimental data make it possible to conclude that the suggested equation for the probability distribution truly describes the qualitative features of this deviation.

The results obtained in this work made it possible, in principle, to state and solve more complicated problems than considered here, such as the description of nonequilibrium turbulent flows with significant deviations of the probability distribution from the normal distribution. Additional difficulties will primarily be associated with computer possibilities. The main qualitative features and mathematical properties of the equation obtained for the velocity probability distribution were analyzed in this work. This analysis is a necessary preliminary step prior to solving more complicated problems. The author is deeply grateful to V. M. Ievlev for positive criticism, useful comments, and support, and to V. R. Kuznetsov for a number of critical comments.

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